# Pair contact process in two dimensions

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We study the stationary properties of the two-dimensional pair contact process, a nonequilibrium lattice model exhibiting a phase transition to an absorbing state with an infinite number of configurations. The critical probability and static critical exponents are determined via Monte Carlo simulations, as well as order-parameter moment ratios and the scaling of the initial density decay. The static critical properties fall in the directed percolation universality class. [S1063-651X(99)11310-2]

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## I. INTRODUCTION

Critical phenomena at absorbing-state phase transitions (i.e., between an active state and one in which the dynamics is frozen), are of longstanding interest in statistical physics, and have enjoyed renewed attention due to connections with epidemics [1], catalytic kinetics [2,3], surface growth [4], self-organized criticality [5–8], and issues of scaling and universality [9,10]. In these systems, conflict between two opposing processes (e.g., creation and annihilation), typically leads to a continuous transition at a critical parameter value. Such transitions are known to fall generically in the universality class of directed percolation (DP) [11–13], although the critical behavior is modified in the presence of local parity conservation [14–17].

Another interesting case (without a conservation law) appears when the dynamics can become trapped in one of an infinite number (in the thermodynamic limit) of absorbing configurations (INAC). Systems of this sort were introduced in catalysis modeling [18–20]; their critical properties have been studied in detail by various researchers [21–28]. In one dimension, the pair contact process (PCP) [22], and other models with INAC exhibit static critical behavior in the DP class [23,29], but the critical exponents associated with the spread of activity from a localized seed are *nonuniversal*, varying continuously with the particle density in the environment [23,28], and follow a generalized hyperscaling relation [24,30]. The anomalous spreading can be traced to a long memory in the dynamics of the order parameter, $\rho$, arising from coupling to an auxiliary field that remains frozen in regions where $\rho = 0$ [25,27,28]. A field theory [i.e., a stochastic partial differential equation for $\rho(\mathbf{x},t)$], incorporating this memory term reproduces the nonuniversal exponents observed in simulations [31].

In two dimensions the situation is much less clear. Simulation results for a microscopic model with INAC [26] conflict with studies of models exhibiting the aforementioned long memory [27,28]. (In particular, it seems possible that

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for one range of densities the spreading dynamics is that of dynamic percolation, while for another range there is compact growth, perhaps with nonuniversal exponents [27], or without well defined scaling behavior [28].) In hopes clarifying the nature of critical spreading in the presence of INAC, we propose to study the pair contact process in two dimensions. Our interest in the PCP is motivated by its simplicity, compared with the model studied in Ref. [26]. The present work is devoted to static critical behavior, and provides the critical parameter value and the ''natural density'' in the absorbing state (defined below) needed for a detailed study of spreading. These results confirm that the static behavior falls in the DP class; analyses of moment ratios and the initial decay of the order parameter provide further support. The balance of this paper is devoted to defining the model and simulation algorithm (Sec. II), simulation results (Sec. III), and a brief summary (Sec. IV).

## II. MODEL

The pair contact process (PCP) is an *interacting particle system*: a Markov process whose state space is a set of particle configurations on a lattice [32,33]. Each nearest-neighbor pair of particles has a rate $p$ of mutual annihilation, and a rate $1-p$ of attempted creation. In a creation attempt on the square lattice, a new particle may appear (with equal likelihood) at any of the six sites neighboring the pair, provided the chosen site is vacant. (Attempts to place a new particle at an occupied site fail.) The PCP exhibits an active phase for $p < p_c$; above this value the system falls into an absorbing configuration that typically contains a substantial density, $\phi$, of particles. (Any arrangement of particles devoid of nearest-neighbor pairs is absorbing.)

In our simulations, we maintain a list of the $N_p$ current nearest-neighbor pairs. At each step we choose a pair at random from the list, and a process (annihilation with probability $p$, creation with probability $1-p$). In case of annihilation, the two particles are simply removed. For creation, we choose a site $\mathbf{x}$ at random from among the six neighbors of the pair, and place a new particle there if $\mathbf{x}$ is currently vacant. (If $\mathbf{x}$ is occupied the configuration remains the same.) The time increment associated with this step is $\Delta t = 1/N_p$, corresponding to one transition per pair per unit time, in agreement with the transition rates that define the process. Following each change we update the list of pairs. We use a
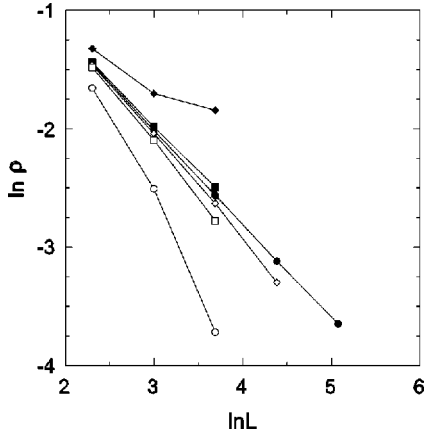
FIG. 1. Stationary pair density vs system size for $p=0.21$, 0.202, 0.201, 0.2005, 0.200, and 0.195 (left to right).

square lattice of $L \times L$ sites, with periodic boundaries; in the studies reported here, all sites are initially occupied.

## III. CRITICAL PROPERTIES

To locate the critical point $p_c$ we study the size-dependence of the (quasi) stationary pair density $\rho$, i.e., the fraction of nearest neighbors harboring a pair of particles, in surviving trials, following a transient period during which $\rho(t)$ relaxes from its initial value of unity. $\rho$ is the order parameter for the PCP, and as such we expect that at the critical point,

$$\rho(p_c,L) \sim L^{-\beta/\nu_\perp}, \tag{1}$$

while off-critical values of $p$ should yield deviations from the power law. We studied the pair density in systems of size $L=10$, 20, 40, 80, and 160, for times $t_m$ ranging from $10^3$ for $L=10$ to $5 \times 10^4$ for $L=160$, with sample sizes ranging from $10^6$ trials ($L=10$) to $10^4$ trials ($L=160$). The results (see Fig. 1) show $\rho(p,L)$ following a power law for $p = 0.2005$, but clearly not for $p=0.200$ or 0.201. These data permit us to conclude that $p_c=0.2005(2)$, the figure in parentheses denoting the uncertainty.

In order to sharpen our estimate for $p_c$, we studied the initial decay of the pair density. We expect the order parameter to decay as a power law, $\rho \sim t^{-\delta}$ in a critical system at short times (i.e., $t < \tau \sim L^{\nu_\parallel/\nu_\perp}$, for which the correlation length $\xi < L$). For off-critical values, we anticipate deviations from a power law when $\xi(t)$ saturates at its stationary value ($< L$), as $\rho(t)$ approaches a nonzero quasistationary density (in the active phase, $p < p_c$), or decays exponentially, for $p > p_c$. In Fig. 2 we show $\rho(t)$ for $p=0.2005$, for three different system sizes, $L=160$, 320, and 640, and observe the expected data collapse at short times. Next we studied values very close to 0.2005 in a system with $L=640$, and evaluated the local slopes of graphs of the sort shown in Fig. 2. [The local slope is obtained from a least-squares linear fit to $\ln\rho$ versus $\ln t$, using sequences of 25 equally spaced points ($\Delta \ln t = 0.1$) to give $\delta(t_a)$, $t_a$ being the geometric mean of the times represented by the data points.] We plot the local slopes versus $t^{-1/2}$ in Fig. 3, and observe that $p_c$ is clearly greater than 0.200 48, and less than 0.200 55. The curve for
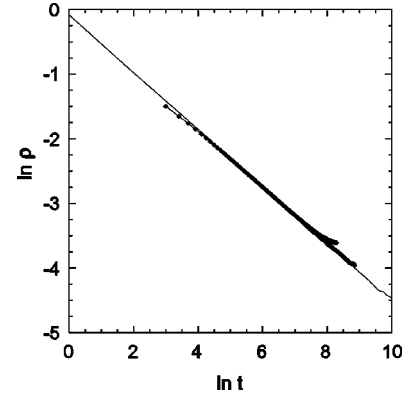


FIG. 2. Decay of the order parameter, $p=0.2005$. +, $L=160$; ●, $L=320$; line, $L=640$.

$p=0.200\,50$ appears to be slightly supercritical, and that for $0.200\,52$ a bit subcritical, leading to the estimate $p_c = 0.200\,51(1)$. The corresponding estimate for $\delta$ (taking into account the uncertainty in $p_c$) is 0.446(6), consistent with recent estimates of $\delta$ for DP in $2+1$ dimensions, which range from 0.4505(10) [34] to 0.452(1) [35].

To estimate the exponent ratio $\beta/\nu_\perp$ we perform a least-squares linear fit to the data shown in Fig. 1; The data for $p=0.2005$ yield the exponent ratio $\beta/\nu_\perp = 0.793(5)$, where the uncertainty represents one standard deviation. The quoted uncertainty, however, does not include the consequences of our uncertainty in $p_c$ itself. In order to see how this would affect our estimate, we ran further studies of the stationary state for $p=0.200\,45$, 0.200 52, and 0.200 55, and found that the apparent value of $\beta/\nu_\perp$ follows a roughly linear trend for this range of values. Our final result, including both intrinsic uncertainties (for a given $p$) and our error estimate for $p_c$, is $\beta/\nu_\perp = 0.792(14)$, in good agreement with the value of 0.799(2) for DP in $2+1$ dimensions [35].

We also determined the survival probability, $P(t,p,L)$, i.e., the probability that the system contains at least one nearest-neighbor pair. For finite $L$, this decays asymptotically as $P \sim e^{-t/\tau_P}$, with the lifetime showing a power-law dependence on the system size at the critical point

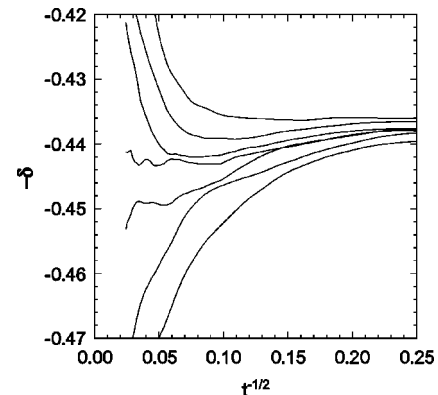$$\tau_P(p_c,L) \sim L^{\nu_\parallel/\nu_\perp}. \tag{2}$$



FIG. 3. Local slope associated with the decay of the order parameter, $L=640$. From top to bottom: $p=0.2004$, 0.200 45, 0.200 48, 0.2005, 0.200 52, 0.200 55, and 0.2006.
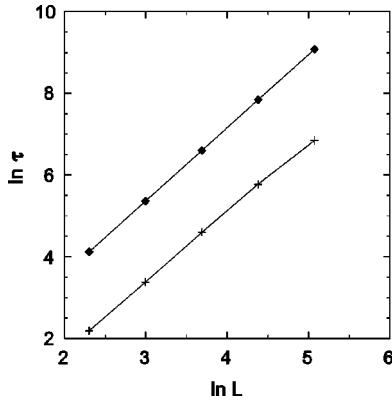
FIG. 4. Relaxation times vs system size at the critical point. Upper set: $\tau_P$, the mean lifetime; lower set: $\tau$, associated with the relaxation of $\rho$ and $\phi$.

The data for $p=0.2005$ are shown in Fig. 4. Following the same procedure as used above for $\beta/\nu_\perp$, we obtain $\nu_\parallel/\nu_\perp = 1.78(3)$. This is again consistent with the DP value, 1.766(2). We find that $\rho$ and the particle density $\phi$ also approach their stationary values exponentially: $\rho(t) - \rho_s \sim e^{-t/\tau}$ (similarly for $\phi$), but on a much shorter time scale than that of $P(t)$: $\tau \simeq \tau_P/10$. Analysis of the data for $\tau$ yields $\nu_\parallel/\nu_\perp = 1.69(5)$. (We note that our data for $\tau$ are of much poorer quality than for $\tau_P$, and so should not be interpreted as indicating non-DP scaling.)

For $p \leq p_c$ the process always falls into the absorbing state. The properties of this state are determined by the probability distribution (induced by the dynamics) on the set of absorbing configurations for system size $L$. Of interest is the particle density $\phi$ in the absorbing state, in particular, the ''natural'' density, defined as the limiting value at the critical point:

$$\phi_{nat} \equiv \lim_{L \to \infty} \phi(p_c, L). \tag{3}$$

(In one dimension, it is only for this particle density that the spreading exponents take DP values [23].) In our simulations at $p_c$ virtually all of the trials end in an absorbing configuration before $t_m$; the final particle density in the absorbing

state yields an estimate for $\phi(p_c, L)$.

Since the dynamics of $\phi$ is tied to that of the order parameter, $\rho$, and since the excess particle density ($\phi(p) - \phi_{nat}$) in a related one-dimensional model is known to be governed by the order-parameter exponent $\beta$ [23], we expect that the leading finite-size correction to the particle density to be $\sim L^{-\beta/\nu_\perp}$, just as for $\rho$. This is confirmed in Fig. 5. Linear fits to the data for $L \geq 40$ yield $\phi \simeq 0.1480 + aL^{-\beta/\nu_\perp}$ and $\rho \simeq bL^{-\beta/\nu_\perp}$, (with $a = 0.9662$ and $b = 1.426$), suggesting that the linear combination $\phi - (a/b)\rho$ will be essentially independent of $L$. This is indeed so for $L \geq 20$, as shown in the inset of Fig. 5, from which we obtain our final estimate, $\phi = 0.1477(1)$.

Order parameter moment ratios provide another tool for assigning a model a universality class; in equilibrium spin systems Binder's reduced fourth cumulant has been widely used for this purpose [36]. A variety of ratios, involving both odd and even moments, have been determined for several one dimensional models with absorbing-state transitions (including the PCP), as well as for the two-dimensional contact process [29]. We determined the stationary order-parameter moments $m_1, \ldots, m_4$ ($m_j \equiv \langle \rho^j \rangle$) in order to evaluate various ratios; the results are listed in Table I. In Fig. 6 we plot several of the moment ratios versus $L^{-1}$; linear fits yield the infinite-$L$ estimates given in Table I. The latter agree quite well with the results for the contact process, providing further support for the PCP belonging to the DP universality class. Curiously, the moment ratios for the PCP appear to approach their limiting values monotonically (with $L$), while the two-dimensional CP exhibits a nonmonotonic $L$ dependence (see Fig. 5 of Ref. [29]).

TABLE I. Ratios of order-parameter moments in the critical PCP. Entries for $L = \infty$ represent linear extrapolations; data for the CP from Ref. [29]. Numbers in parentheses denote uncertainties in the last figure.

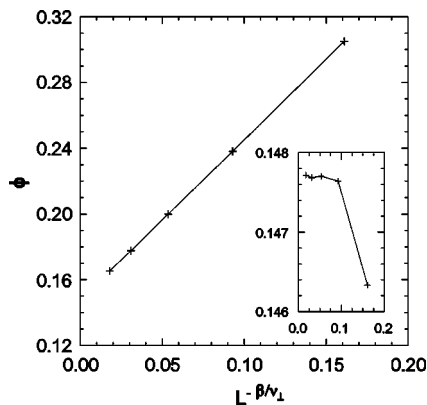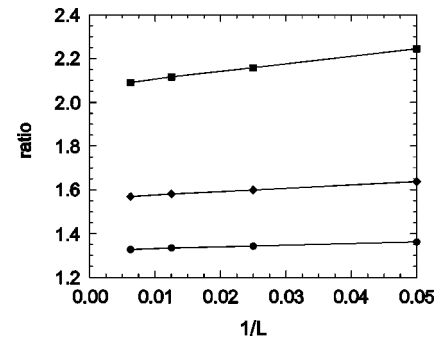| $L$ | $m_2/m_1^2$ | $m_3/m_1^3$ | $m_3/(m_1 m_2)$ | $m_4/m_2^2$ |
|------|------------|------------|-----------------|-------------|
| 20 | 1.362(3) | 2.300(6) | 1.638(5) | 2.247(10) |
| 40 | 1.343(3) | 2.147(7) | 1.599(5) | 2.159(9) |
| 80 | 1.334(3) | 2.111(9) | 1.582(6) | 2.116(10) |
| 160 | 1.327(4) | 2.086(9) | 1.571(8) | 2.093(13) |
| $\infty$ | 1.323(3) | 2.067(9) | 1.56(1) | 2.07(1) |
| CP | 1.326(1) | 2.080(1) | 1.569(1) | 2.093(8) |



FIG. 5. Particle density in absorbing configurations at the critical point vs $L^{-\beta/\nu_\perp}$. The inset is a plot of $\phi - 0.6776\rho$.



FIG. 6. Order-parameter moment ratios in the critical PCP. Upper set, $m_4/m_2^2$; middle set, $m_3/(m_1 m_2)$; lower set, $m_2/m_1^2$.

## IV. SUMMARY

We studied the stationary critical properties of the pair contact process in two dimensions. On the basis of the exponent ratios $\beta/\nu_\perp$ and $\nu_\parallel/\nu_\perp$, moment ratios, and the initial decay of the order parameter, we can assign the PCP to the directed percolation universality class, generic for absorbing-state transitions without a conservation law or special sym-metry. Our best estimate for the critical parameter is $p_c = 0.200\,51(1)$. The issue of spreading dynamics will be addressed in future work.

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